

The near-Hopf ring structure on the integral cohomology of 1-connected Lie groups

Haibao Duan

Institute of Mathematics, Chinese Academy of Sciences
dhb@math.ac.cn

Abstract

Let G be an exceptional Lie group with a maximal torus T . Based on Schubert calculus on the flag manifold G/T we have described the integral cohomology ring $H^*(G)$ by explicitly constructed generators in [4], and determined the structure of $H^*(G; \mathbb{F}_p)$ as a Hopf algebra over the Steenrod algebra in [5]. Combining these works we determine the near-Hopf ring structure on $H^*(G)$.

2000 *Mathematical Subject Classification*: 55T10

Key words and phrases: Lie groups; cohomology; Hopf algebra

1 Introduction

Let G be a compact 1-connected Lie group with multiplication $\mu : G \times G \rightarrow G$ and integral cohomology ring $H^*(G)$. The induced ring map

$$(1.1) \quad \mu^* : H^*(G) \rightarrow H^*(G \times G)$$

will be referred to as the *near-Hopf ring structure* on $H^*(G)$. In [4, Theorem 6] we have described the rings $H^*(G)$ for all exceptional Lie groups G by explicitly constructed *primary generators*. In this sequel to [4] we determine the action of μ^* on $H^*(G)$ with respect to these generators.

We shall start with a brief introduction in §2 on two sets of generators for $H^*(G)$; $\varrho_{i_1}, \dots, \varrho_{i_n}$ and \mathcal{C}_I with I a certain multi-index, constructed explicitly in [4]. The elements ϱ_i have infinite order and their square free products form a \mathbb{Z} -basis for the free part of $H^*(G)$. The \mathcal{C}_I come from the Bockstein of a distinguished set of elements in $H^*(G; \mathbb{F}_p)$ with $p = 2, 3$ or 5 .

For a prime p the multiplication μ induces also the *co-product*

$$(1.2) \quad \mu_p^* : H^*(G; \mathbb{F}_p) \rightarrow H^*(G; \mathbb{F}_p) \otimes H^*(G; \mathbb{F}_p)$$

on the algebra $H^*(G; \mathbb{F}_p)$ by virtue of the Kunneth formula. It furnishes $H^*(G; \mathbb{F}_p)$ with the structure of a *Hopf algebra*. In §3 we establish a "*pull-back formula*" that reduces the computation of $\mu^*(z)$ to that of $\mu_p^*(z \bmod p)$ for a generators $z = \varrho_i$ or \mathcal{C}_I , and deduce from [5, Theorem 2; Theorem 4.1] a presentation of the Hopf algebra $H^*(G; \mathbb{F}_p)$.

With these preparation the near-Hopf ring $H^*(G)$ for an exceptional G is obtained and presented in Theorems 1–5 in §4.

The author feels grateful to M. Mimura for encouraging him to work out these results. Earlier in the 1950's Borel calculated the integral cohomology ring $H^*(G)$ for $G = G_2, F_4$ [1, 2]. However, the action of μ^* was absent even in these cases. On the other hand, combining our results with previous knowledge for the classical groups $G = SU(n), Sp(n), Spin(n)$ by Borel and Pitties [2, 9] completes the project to determine the near-Hopf ring structure of all compact 1-connected Lie groups.

2 The generators for the ring $H^*(G)$

Assume that G is a compact 1-connected Lie group with a maximal torus $T \subset G$, and let \mathbb{F} be either the ring \mathbb{Z} of integers, the field \mathbb{Q} of rationals, or one of the finite fields \mathbb{F}_p . In view of the Leray-Serre spectral sequence $\{E_r^{*,*}(G; \mathbb{F}), d_r\}$ of the fibration

$$(2.1) \quad T \rightarrow G \xrightarrow{\pi} G/T$$

with coefficients in \mathbb{F} , we recall from [4] the two sets ϱ_i 's and \mathcal{C}_I 's of generators for $H^*(G)$ mentioned in §1.

The next result was due to Leray [7] for $\mathbb{F} = \mathbb{Q}$, extended to $\mathbb{F} = \mathbb{F}_p$ by Serre [10], conjectured for $\mathbb{F} = \mathbb{Z}$ by Kač [6] and Marlin [8].

Lemma 2.1 ([4, Remark 5.3]). As graded groups $H^*(G; \mathbb{F}) = E_3^{*,*}(G; \mathbb{F})$. \square

Lemma 2.1 grants us with the ready-made decomposition

$$(2.2) \quad H^*(G; \mathbb{F}) = E_3^{*,0}(G; \mathbb{F}) \oplus E_3^{*,1}(G; \mathbb{F}) \oplus \cdots \oplus E_3^{*,n}(G; \mathbb{F}).$$

Concrete presentations for the lower terms $E_3^{*,0}(G; \mathbb{Z})$ and $E_3^{*,1}(G; \mathbb{Z})$ bring us certain elements in $H^*(G)$ that suffice to generate the ring $H^*(G)$. We start by introducing for each exceptional G a set of *special Schubert classes* $y_i \in H^i(G/T)$ on G/T by their *Weyl coordinates* [3] in the table below:

G/T	G_2/T	F_4/T	$E_n/T, n = 6, 7, 8$
y_6	$\sigma_{[1,2,1]}$	$\sigma_{[3,2,1]}$	$\sigma_{[5,4,2]}, n = 6, 7, 8$
y_8			$\sigma_{[6,5,4,2]}, n = 6, 7, 8$
y_{10}			$\sigma_{[7,6,5,4,2]}, n = 7, 8$
y_{12}			$\sigma_{[1,3,6,5,4,2]}, n = 8$
y_{18}			$\sigma_{[1,5,4,3,7,6,5,4,2]}, n = 7, 8$
y_{20}			$\sigma_{[1,6,5,4,3,7,6,5,4,2]}, n = 8$
y_{30}			$\sigma_{[5,4,2,3,1,6,5,4,3,8,7,6,5,4,2]}, n = 8$

and set $x_i := \pi^*(y_i) \in H^*(G)$. The next result was shown in [4, Lemma 3.1].

Lemma 2.2. The subring $E_3^{*,0}(G; \mathbb{Z}) \subset H^*(G)$ has the presentation

$$\begin{aligned} E_3^{*,0}(G_2; \mathbb{Z}) &= \mathbb{Z}[x_6] / \langle 2x_6, x_6^2 \rangle; \\ E_3^{*,0}(F_4; \mathbb{Z}) &= \mathbb{Z}[x_6, x_8] / \langle 2x_6, x_6^2, 3x_8, x_8^3 \rangle; \\ E_3^{*,0}(E_6; \mathbb{Z}) &= \mathbb{Z}[x_6, x_8] / \langle 2x_6, x_6^2, 3x_8, x_8^3 \rangle; \end{aligned}$$

$$\begin{aligned}
E_3^{*,0}(E_7; \mathbb{Z}) &= \mathbb{Z}[x_6, x_8, x_{10}, x_{18}] / \langle 2x_6, 3x_8, 2x_{10}, 2x_{18}, x_6^2, x_8^3, x_{10}^2, x_{18}^2 \rangle; \\
E_3^{*,0}(E_8; \mathbb{Z}) &= \mathbb{Z}[x_6, x_8, x_{10}, x_{12}, x_{18}, x_{20}, x_{30}] / \\
&\quad \langle 2x_6, 3x_8, 2x_{10}, 5x_{12}, 2x_{18}, 3x_{20}, 2x_{30}, x_6^8, x_8^3, x_{10}^4, x_{12}^5, x_{18}^2, x_{20}^3, x_{30}^2 \rangle. \square
\end{aligned}$$

Let (G, p) be a pair with G an exceptional Lie group and $H^*(G)$ containing non-trivial p -torsion, and let $r(G, p)$ be the set of the degrees of the basic “generalized Weyl invariants” of G over \mathbb{F}_p . Precisely we have (see [6])

$$(2.4) \quad \begin{array}{c|c} (G, p) & e(G, p) \subset r(G, p) \\ \hline (G_2, 2) & \{2, \underline{3}\} \\ (F_4, 2) & \{2, \underline{3}, 8, 12\} \\ (E_6, 2) & \{2, \underline{3}, 5, 8, 9, 12\} \\ (E_7, 2) & \{2, \underline{3}, \underline{5}, 8, \underline{9}, 12, 14\} \\ (E_8, 2) & \{2, \underline{3}, \underline{5}, 8, \underline{9}, 12, 14, \underline{15}\} \\ (F_4, 3) & \{2, \underline{4}, 6, 8\} \\ (E_6, 3) & \{2, \underline{4}, 5, 6, 8, 9\} \\ (E_7, 3) & \{2, \underline{4}, 6, 8, 10, 14, 18\} \\ (E_8, 3) & \{2, \underline{4}, 8, \underline{10}, 14, 18, 20, 24\} \\ (E_8, 5) & \{2, \underline{6}, 8, 12, 14, 18, 20, 24\} \end{array} .$$

Let $e(G, p)$ be the subset of $r(G, p)$ whose elements are underlined. In [DZ₂, (2.10)] we have constructed a subset $\{\zeta_{2s-1} \in E_3^{2s-2,1}(G; \mathbb{F}_p) \mid s \in r(G, p)\}$, whose elements are called p -primary generators on $H^*(G; \mathbb{F}_p)$, that satisfy the following properties ([4, Theorem 1]).

Lemma 2.3. *One has the additive presentation*

$$H^*(G; \mathbb{F}_p) = E_3^{*,0}(G_2; \mathbb{F}_p) \otimes \Delta(\zeta_{2s-1})_{s \in r(G, p)}$$

on which the Bockstein $\delta_p : H^r(G; \mathbb{F}_p) \rightarrow H^{r+1}(G)$ is given by

$$\delta_p(\zeta_{2s-1}) = \begin{cases} -x_{2s} & \text{if } s \in e(G, p); \\ 0 & \text{if } s \notin e(G, p), \end{cases}$$

where $E_3^{*,0}(G; \mathbb{F}_p) = E_3^{*,0}(G; \mathbb{Z}) \otimes \mathbb{F}_p$, and where $\Delta(\zeta_{2s-1})_{s \in r(G, p)}$ is the \mathbb{F}_p -space with basis the square free monomials in $\{\zeta_{2s-1} \mid s \in r(G, p)\}$. \square

Definition 2.4. For a subset $I \subseteq e(G, p)$ we put $\mathcal{C}_I := \delta_p(\zeta_I) \in H^*(G)$, where $\zeta_I = \prod_{s \in I} \zeta_{2s-1} \in E_3^{*,k}(G; \mathbb{F}_p)$ with k the cardinality of I . \square

For a presentation of $E_3^{*,1}(G; \mathbb{Z})$ we note that the cup product in $H^*(G)$ defines an action of $E_3^{*,0}(G; \mathbb{Z})$ on $E_3^{*,1}(G; \mathbb{Z})$. Given a ring A and a finite set $\{u_1, \dots, u_t\}$ write $A\{u_i\}_{1 \leq i \leq t}$ for the free A -module with basis $\{u_1, \dots, u_t\}$. For an exceptional G with rank n let $\mathcal{D}_G = \{d_1, \dots, d_n\}$ be the set of degrees of basic W -invariants of G over \mathbb{Q} . As is classical we have (e.g. [6])

$$(2.5) \quad \begin{array}{c|c} \text{Type of } G & \mathcal{D}_G \\ \hline G_2 & \{2, 6\} \\ F_4 & \{2, 3, 8, 12\} \\ E_6 & \{2, 5, 6, 8, 9, 12\} \\ E_7 & \{2, 6, 8, 10, 12, 14, 18\} \\ E_8 & \{2, 8, 12, 14, 18, 20, 24, 30\} \end{array}$$

The next result is indicated in the proof of [4, Theorem 2].

Lemma 2.5. *For each $i \in \mathcal{D}_G$ there is an element $\varrho_{2i-1} \in E_3^{2i-2,1}(G; \mathbb{Z})$ of infinite order so that*

$$E_3^{*,1}(G; \mathbb{Z}) = E_3^{*,0}(G; \mathbb{Z})\{\varrho_{2i-1}\}_{i \in \mathcal{D}_G}.$$

Moreover, a set of ring generators for $H^*(G)$ is

$$(2.6) \quad \mathcal{G}(G) = \{\varrho_{2i-1}, C_I \mid i \in \mathcal{D}_G, I \subseteq e(G, p), p = 2, 3, 5\}. \square$$

Remark 2.6. In (2.3) the Weyl coordinate of a Schubert class $y_i \in H^i(G/T)$ allows one to construct explicitly the Schubert variety on G/T Kronecker dual to y_i [3].

The subring $E_3^{*,0}(G; \mathbb{Z}) \subset H^*(G)$ coincides with the Chow ring of the reductive algebraic group G^c corresponding to G , see [4, Lemma 3.1]. \square

3 The pull-back formula

For convenience we fashion from μ^* the *reduced co-product*

$$\psi : H^*(G) \rightarrow H^*(G \times G)$$

by $\psi(x) = \mu^*(x) - (x \otimes 1 + 1 \otimes x)$. Its \mathbb{F}_p -analogue (resp. \mathbb{Q} -analogue) is denoted by ψ_p (resp. ψ_0). The next result contains a formula that reduces calculation of ψ to that of ψ_p .

For a topological space X let $\tau(X)$ be the torsion ideal in the integral cohomology $H^*(X)$. For a prime p write $\tau_p(X)$ be the p -primary component of $\tau(X)$. The mod p reduction $H^r(X) \rightarrow H^r(X; \mathbb{F}_p)$ is denoted by r_p .

Lemma 3.1. *For an exceptional Lie group G let $\mathcal{G}(G)$ be the set of generators for $H^*(G)$ given in (2.5). Then*

$$a) \quad \psi(x) \in \tau(G \times G) \text{ for all } x \in \mathcal{G}(G).$$

Moreover, for $K = G \times G$ or G

$$b) \quad \tau(K) = \oplus_{p=2,3,5} \tau_p(K);$$

$$c) \quad \text{the reduction } r_p \text{ restricts to an injection } \tau_p(K) \rightarrow H^*(K; \mathbb{F}_p),$$

In particular, for all $z \in \mathcal{G}(G)$ one has

$$(3.1) \quad \psi(z) = r_2^{-1} \psi_2(r_2(z)) + r_3^{-1} \psi_3(r_3(z)) + r_5^{-1} \psi_5(r_5(z)).$$

Proof. For $x = \mathcal{C}_I$ assertion a) follows from $x \in \tau(G)$. Assume now that $x = \varrho_j \in \mathcal{G}(G)$. The map $i : H^*(G) \rightarrow H^*(G; \mathbb{Q})$ induced by the inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$ of coefficients clearly preserves the decomposition (2.2) and therefore, restricts to a homomorphism

$$i : E_3^{*,1}(G; \mathbb{Z}) = E_3^{*,0}(G; \mathbb{Z})\{\varrho_{2i-1}\}_{i \in \mathcal{D}_G} \rightarrow E_3^{*,1}(G; \mathbb{Q}).$$

Since the subset $E_3^{*,1}(G; \mathbb{Q}) \subset H^*(G; \mathbb{Q})$ consists of primitive elements [7], we get $\psi(\varrho_j) \in \tau(G \times G)$ from $\psi_0(i(\varrho_j)) = 0$. This shows a).

According to [4, Theorem 2] results in b) and c) hold for any 1-connected Lie group K . Finally, the commutative diagram

$$\begin{array}{ccc} H^*(G) & \xrightarrow{r_p} & H^*(G; \mathbb{F}_p) \\ \mu^* \downarrow & & \mu_p^* \downarrow \\ H^*(G \times G) & \xrightarrow{r_p} & H^*(G \times G; \mathbb{F}_p) \end{array}$$

induced by μ implies that $\text{Im } \psi_p \circ r_p = \text{Im } r_p \circ \psi \subset H^*(G \times G; \mathbb{F}_p)$. One obtains (3.1) from a), b) and c), where the validity of r_p^{-1} , $p = 2, 3, 5$, follows from the injectivity of $r_p : \tau_p(G \times G) \rightarrow H^*(G \times G; \mathbb{F}_p)$ by c). \square

In order to apply formula (3.1) to determine the action of ψ we need accounts for

- i) the reduction $r_p : H^*(G) \rightarrow H^*(G; \mathbb{F}_p)$ with respect to the set $\mathcal{G}(G)$ of generators for $H^*(G)$ and the presentation of $H^*(G; \mathbb{F}_p)$ in Lemma 2.3;
- ii) the Hopf algebra structure on $H^*(G; \mathbb{F}_p)$ with respect to its presentation in Lemma 2.3.

For i) we have the next result from [4, Lemma 3.5].

Lemma 3.2. *With respect to the set $\mathcal{G}(G)$ of generators on $H^*(G)$ and the set $\{\zeta_{2s-1}\}_{s \in r(G,p)}$ of p -primary generators on $H^*(G; \mathbb{F}_p)$, the mod p reduction $r_p : H^*(G) \rightarrow H^*(G; \mathbb{F}_p)$ is given by*

$$(3.2) \quad \begin{array}{c|ccc} & p=2 & p=3 & p=5 \\ \hline r_p(\varrho_3) & \zeta_3 & \zeta_3 & \zeta_3 \\ \hline r_p(\varrho_9) & \zeta_9 & \zeta_9 & \zeta_9 \\ \hline r_p(\varrho_{11}) & x_6 \zeta_5 & -\zeta_{11} & 2\zeta_{11} \\ \hline r_p(\varrho_{15}) & \zeta_{15} & \zeta_{15} & \zeta_{15} \\ \hline r_p(\varrho_{17}) & \zeta_{17} & \zeta_{17} & \zeta_{17} \\ \hline r_p(\varrho_{19}) & x_{10} \zeta_9 & -\zeta_{19} & 2\zeta_{19} \\ \hline r_p(\varrho_{23}) & \zeta_{23} & -x_8^2 \zeta_7 & 2\zeta_{23} \\ \hline r_p(\varrho_{27}) & \zeta_{27} & \zeta_{27} & \zeta_{27} \\ \hline r_p(\varrho_{35}) & x_{18} \zeta_{17} & -\zeta_{35} & 2\zeta_{35} \\ \hline r_p(\varrho_{39}) & x_{10}^3 \zeta_9 & -\zeta_{39} & 2\zeta_{39} \\ \hline r_p(\varrho_{47}) & x_6^7 \zeta_5 & -\zeta_{47} & 2\zeta_{47} \\ \hline r_p(\varrho_{59}) & x_{30} \zeta_{29} & -x_{20}^2 \zeta_{19} & 2x_{12}^4 \zeta_{11} \end{array} ;$$

$$(3.3) \quad r_p(\mathcal{C}_I) = \beta_p(\zeta_I), \quad I \subseteq e(G; p),$$

where $\beta_p = r_p \circ \delta_p$ is the Bockstein operator on $H^*(G; \mathbb{F}_p)$. \square

With respect to the presentation of $H^*(G; \mathbb{F}_p)$ in Lemma 2.3, the Hopf algebra structure on $H^*(G; \mathbb{F}_p)$ has been determined, see [5, Remark 4.6]. For simplicity we reserve x_i for $r_p(x_i)$.

Lemma 3.3. *Let (G, p) be a pair with G exceptional and $H^*(G)$ containing non-trivial p -torsion.*

(3.4) The algebra $H^*(G; \mathbb{F}_2)$ has the presentation

$$\begin{aligned}
H^*(G_2; \mathbb{F}_2) &= \mathbb{F}_2[x_6] / \langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\zeta_3) \otimes \Lambda_{\mathbb{F}_2}(\zeta_5); \\
H^*(F_4; \mathbb{F}_2) &= \mathbb{F}_2[x_6] / \langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\zeta_3) \otimes \Lambda_{\mathbb{F}_2}(\zeta_5, \zeta_{15}, \zeta_{23}); \\
H^*(E_6; \mathbb{F}_2) &= \mathbb{F}_2[x_6] / \langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\zeta_3) \otimes \Lambda_{\mathbb{F}_2}(\zeta_5, \zeta_9, \zeta_{15}, \zeta_{17}, \zeta_{23}); \\
H^*(E_7; \mathbb{F}_2) &= \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}]}{\langle x_6^2, x_{10}^2, x_{18}^2 \rangle} \otimes \Delta_{\mathbb{F}_2}(\zeta_3, \zeta_5, \zeta_9) \otimes \Lambda_{\mathbb{F}_2}(\zeta_{15}, \zeta_{17}, \zeta_{23}, \zeta_{27}); \\
H^*(E_8; \mathbb{F}_2) &= \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}, x_{30}]}{\langle x_6^2, x_{10}^4, x_{18}^2, x_{30}^2 \rangle} \otimes \Delta_{\mathbb{F}_2}(\zeta_3, \zeta_5, \zeta_9, \zeta_{15}, \zeta_{23}) \otimes \Lambda_{\mathbb{F}_2}(\zeta_{17}, \zeta_{27}, \zeta_{29}),
\end{aligned}$$

where

$$\begin{aligned}
\zeta_3^2 &= x_6 \text{ in } G_2, F_4, E_6, E_7, E_8; \\
\zeta_5^2 &= x_{10}, \quad \zeta_9^2 = x_{18} \text{ in } E_7, E_8; \\
\zeta_{15}^2 &= x_{30}, \quad \zeta_{23}^2 = x_6^6 x_{10} \text{ in } E_8,
\end{aligned}$$

on which the action of ψ_2 is given by

$$\begin{aligned}
\psi_2(\zeta_i) &= 0 \text{ if } i = 3, 5, 9, 17, \text{ or } 15, 23 \text{ for } F_4; \\
\psi_2(\zeta_i) &= x_6 \otimes \zeta_{i-6} \text{ if } i = 15, 23 \text{ for } E_6; \\
\psi_2(\zeta_{15}) &= \beta_2(\zeta_9 \otimes \zeta_5) \text{ for } E_7; \\
\psi_2(\zeta_{23}) &= \beta_2(\zeta_{17} \otimes \zeta_5) \text{ for } E_7; \\
\psi_2(\zeta_{27}) &= \beta_2(\zeta_{17} \otimes \zeta_9) \text{ for } E_7; \\
\psi_2(\zeta_{15}) &= \beta_2(\zeta_9 \otimes \zeta_5) + x_6^2 \otimes \zeta_3 \text{ for } E_8; \\
\psi_2(\zeta_{23}) &= \beta_2(\zeta_{17} \otimes \zeta_5) + \sum_{s+t=2} x_6^s \otimes x_6^t \beta_2(\zeta_5 \otimes \zeta_5) + x_{10}^2 \otimes \zeta_3 \text{ for } E_8; \\
\psi_2(\zeta_{27}) &= \beta_2(\zeta_{17} \otimes \zeta_9) + x_6^4 \otimes \zeta_3 \text{ for } E_8; \\
\psi_2(\zeta_{29}) &= x_{10}^2 \otimes \zeta_9 + \zeta_{17} \otimes x_6^2 + x_6^4 \otimes \zeta_5 \text{ for } E_8.
\end{aligned}$$

(3.5) The algebra $H^*(G; \mathbb{F}_3)$ has the presentation

$$\begin{aligned}
H^*(G_2; \mathbb{F}_3) &= \Lambda_{\mathbb{F}_3}(\zeta_3, \zeta_{11}); \\
H^*(F_4; \mathbb{F}_3) &= \mathbb{F}_3[x_8] / \langle x_8^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\zeta_3, \zeta_7, \zeta_{11}, \zeta_{15}); \\
H^*(E_6; \mathbb{F}_3) &= \mathbb{F}_3[x_8] / \langle x_8^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\zeta_3, \zeta_7, \zeta_9, \zeta_{11}, \zeta_{15}, \zeta_{17}); \\
H^*(E_7; \mathbb{F}_3) &= \mathbb{F}_3[x_8] / \langle x_8^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\zeta_3, \zeta_7, \zeta_{11}, \zeta_{15}, \zeta_{19}, \zeta_{27}, \zeta_{35}); \\
H^*(E_8; \mathbb{F}_3) &= \mathbb{F}_3[x_8, x_{20}] / \langle x_8^3, x_{20}^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\zeta_3, \zeta_7, \zeta_{15}, \zeta_{19}, \zeta_{27}, \zeta_{35}, \zeta_{39}, \zeta_{47})
\end{aligned}$$

on which the action of ψ_3 is given by

$$\begin{aligned}
\psi_3(\zeta_i) &= 0 \text{ if } i = 3, 7, 9, 17, 19; \\
\psi_3(\zeta_{11}) &= -x_8 \otimes \zeta_3; \\
\psi_3(\zeta_{15}) &= -\beta_3(\zeta_7 \otimes \zeta_7); \\
\psi_3(\zeta_{27}) &= -\beta_3(\zeta_7 \otimes \zeta_{19}) \text{ for } E_7; \\
\psi_3(\zeta_{35}) &= \zeta_{27} \otimes x_8 - x_8 \otimes \zeta_{27} - x_8 \otimes x_8 \zeta_{19} \text{ for } E_7; \\
\psi_3(\zeta_{27}) &= \beta_3(\zeta_{19} \otimes \zeta_7) \text{ for } E_8; \\
\psi_3(\zeta_{35}) &= \zeta_{27} \otimes x_8 - x_8 \otimes \zeta_{27} - x_{20} \otimes \zeta_{15} - \beta_3(x_8 \zeta_{19} \otimes \zeta_7) \text{ for } E_8; \\
\psi_3(\zeta_{39}) &= \beta_3(\zeta_{19} \otimes \zeta_{19}); \\
\psi_3(\zeta_{47}) &= x_{20} \otimes \zeta_{27} - \zeta_{39} \otimes x_8 - \beta_3(x_{20} \zeta_{19} \otimes \zeta_7).
\end{aligned}$$

(3.6) The algebra $H^*(E_8; \mathbb{F}_5)$ has the presentation

$$H^*(E_8; \mathbb{F}_5) = \mathbb{F}_5[x_{12}] / \langle x_{12}^5 \rangle \otimes \Lambda(\zeta_3, \zeta_{11}, \zeta_{15}, \zeta_{23}, \zeta_{27}, \zeta_{35}, \zeta_{39}, \zeta_{47})$$

on which the action of ψ_5 is given by

$$\begin{aligned} \psi_5(\zeta_i) &= 0, \quad i = 3, 11, \\ \psi_5(\zeta_{15}) &= x_{12} \otimes \zeta_3, \\ \psi_5(\zeta_{23}) &= 2\beta_5(\zeta_{11} \otimes \zeta_{11}), \\ \psi_5(\zeta_{27}) &= -x_{12} \otimes \zeta_{15} + 2x_{12}^2 \otimes \zeta_3, \\ \psi_5(\zeta_{35}) &= x_{12} \otimes \zeta_{23} + \beta_5(3x_{12}\zeta_{11} \otimes \zeta_{11} - \zeta_{11} \otimes \zeta_{11}x_{12}), \\ \psi_5(\zeta_{39}) &= 3x_{12} \otimes \zeta_{27} + x_{12}^2 \otimes \zeta_{15} + 2x_{12}^3 \otimes \zeta_3 \\ \psi_5(\zeta_{47}) &= x_{12} \otimes \zeta_{35} - 2x_{12}^2 \otimes \zeta_{23} + \beta_5(\zeta_{11} \otimes x_{12}^2\zeta_{11} \\ &\quad + 3x_{12}\zeta_{11} \otimes x_{12}\zeta_{11} + 3x_{12}^2\zeta_{11} \otimes \zeta_{11}). \square \end{aligned}$$

Remark 3.4. We emphasize that the ring $H^*(G)$ (resp. the algebra $H^*(G; \mathbb{F}_p)$) admits many sets of generators subject to a given degree constraints, and the actions of r_p (resp. ψ_p) vary sensitively with respect to different choices of a set of generators. However, in the context of [4] the elements $\varrho_{2i-1} \in E_3^{*,1}(G; \mathbb{Z})$ and $\zeta_{2s-1} \in E_3^{*,1}(G; \mathbb{F}_p)$ are coherently constructed from explicit polynomials in certain Schubert classes on G/T . Consequently, with respect to them the presentations of r_p in Lemma 3.2 (resp. ψ_p in Lemma 3.3) were derived by computing with these polynomials.

4 The results

Historically, the Hopf algebras $H^*(G; \mathbb{F}_p)$ for exceptional Lie groups G have been studied by many authors, notably, by Borel, Araki, Toda, Kono, Mimura, Shimada, see [5] for an account for the history. However, these results can not be directly used in the place of Lemma 3.3 since they were obtained by various methods, presented by generators with different origins, using case by case computations depending on G and p and without referring to the integral cohomology $H^*(G)$.

In comparison, with our generators ϱ_j 's and ζ_i 's stemming solely from certain polynomials in the Schubert classes on G/T in [4], the relationships between $H^*(G)$ and $H^*(G; \mathbb{F}_p)$ are transparent for all prime p , as indicated by Lemma 3.2. For this reason the pull-back formula is directly applicable to translate the Hopf algebra structure on $H^*(G; \mathbb{F}_p)$ to the near-Hopf ring structure on $H^*(G)$.

An element $x \in H^*(G)$ is called *primitive* if $\psi(x) = 0$. Let $\mathcal{P}(G)$ be the graded \mathbb{Z} -module of all primitive elements in $H^*(G)$.

In Theorems 1–5 below the near-Hopf rings $H^*(G)$ for all exceptional G are presented in terms of the set $\mathcal{G}(G)$ of generators specified in (2.6). In view of Definition 2.4 for the class $\mathcal{C}_I \in \mathcal{G}(G)$ we note that

- i) $\mathcal{C}_I = x_i$ for $I = \{i\} \subseteq e(G, p)$ a singleton;
- ii) $\deg \mathcal{C}_I = 2(i_1 + \cdots + i_k) - k + 1$ if $I = (i_1, \dots, i_k) \subseteq e(G, p)$;
- iii) the value of $\psi(\mathcal{C}_I)$, $I \subseteq e(G, p)$, is determined by $\psi_p(\zeta_{2s-1})$, $s \in e(G, p)$ because of the relation $\delta_p \circ \psi_p = \psi \circ \delta_p$.

Theorem 1. *With respect to the ring presentation*

$$H^*(G_2) = \Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_{11}) \oplus \tau_2(G_2)$$

where

$$\tau_2(G_2) = F_2[x_6]^+ / \langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\varrho_3),$$

and where $\varrho_3^2 = x_6$, $x_6\varrho_{11} = 0$, the reduced co-product ψ is given by

$$\begin{aligned} \{\varrho_3, x_6\} &\subset \mathcal{P}(G_2); \\ \psi(\varrho_{11}) &= \delta_2(\zeta_5 \otimes \zeta_5). \end{aligned}$$

Theorem 2. *With respect to the ring presentation*

$$H^*(F_4) = \Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_{11}, \varrho_{15}, \varrho_{23}) \oplus \tau_2(F_4) \oplus \tau_3(F_4)$$

where

$$\begin{aligned} \tau_2(F_4) &= F_2[x_6]^+ / \langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\varrho_3) \otimes \Lambda_{\mathbb{F}_2}(\varrho_{15}, \varrho_{23}), \\ \tau_3(F_4) &= F_3[x_8]^+ / \langle x_8^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_{11}, \varrho_{15}), \end{aligned}$$

and where $\varrho_3^2 = x_6$, $x_6\varrho_{11} = 0$, $x_8\varrho_{23} = 0$, the reduced co-product ψ is given by

$$\begin{aligned} \{\varrho_3, x_6, x_8\} &\subset \mathcal{P}(F_4); \\ \psi(\varrho_{11}) &= \delta_2(\zeta_5 \otimes \zeta_5) + x_8 \otimes \varrho_3; \\ \psi(\varrho_{15}) &= -\delta_3(\zeta_7 \otimes \zeta_7) \\ \psi(\varrho_{23}) &= \delta_3(\zeta_7 \otimes \zeta_7 x_8 - \zeta_7 x_8 \otimes \zeta_7). \end{aligned}$$

Theorem 3. *With respect to the ring presentation*

$$H^*(E_6) = \Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_9, \varrho_{11}, \varrho_{15}, \varrho_{17}, \varrho_{23}) \oplus \tau_2(E_6) \oplus \tau_3(E_6)$$

where

$$\begin{aligned} \tau_2(E_6) &= F_2[x_6]^+ / \langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\varrho_3) \otimes \Lambda_{\mathbb{F}_2}(\varrho_9, \varrho_{15}, \varrho_{17}, \varrho_{23}), \\ \tau_3(E_6) &= F_3[x_8]^+ / \langle x_8^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_9, \varrho_{11}, \varrho_{15}, \varrho_{17}), \end{aligned}$$

and where $\varrho_3^2 = x_6$, $x_6\varrho_{11} = 0$, $x_8\varrho_{23} = 0$, the reduced co-product ψ is given by

$$\begin{aligned} \{\varrho_3, \varrho_9, \varrho_{17}, x_6, x_8\} &\subset \mathcal{P}(E_6); \\ \psi(\varrho_{11}) &= \delta_2(\zeta_5 \otimes \zeta_5) + x_8 \otimes \varrho_3; \\ \psi(\varrho_{15}) &= x_6 \otimes \varrho_9 - \delta_3(\zeta_7 \otimes \zeta_7); \\ \psi(\varrho_{23}) &= x_6 \otimes \varrho_{17} + \delta_3(\zeta_7 x_8 \otimes \zeta_7 - \zeta_7 \otimes \zeta_7 x_8). \end{aligned}$$

Theorem 4. *With respect to the ring presentation*

$$H^*(E_7) = \Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_{11}, \varrho_{15}, \varrho_{19}, \varrho_{23}, \varrho_{27}, \varrho_{35}) \oplus \tau_2(E_7) \oplus \tau_3(E_7)$$

where

$$\begin{aligned}\tau_2(E_7) &= \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}, \mathcal{C}_I]^+}{\langle x_6^2, x_{10}^2, x_{18}^2, \mathcal{D}_J, \mathcal{R}_K, \mathcal{S}_{I,J}, \mathcal{H}_{t,L} \rangle} \otimes \Delta_{\mathbb{F}_2}(\varrho_3) \otimes \Lambda_{\mathbb{F}_2}(\varrho_{15}, \varrho_{23}, \varrho_{27}) \\ &\quad \text{with } t \in e(E_7, 2) = \{3, 5, 9\}, I, J, L \subseteq e(E_7, 2), |I|, |J| \geq 2; \\ \tau_3(E_7) &= \frac{\mathbb{F}_3[x_8]^+}{\langle x_8^3 \rangle} \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_{11}, \varrho_{15}, \varrho_{19}, \varrho_{27}, \varrho_{35}),\end{aligned}$$

and where $\varrho_3^2 = x_6$, $x_8 \varrho_{23} = 0$, the reduced co-product ψ is given by

$$\begin{aligned}\{\varrho_3, x_6, x_8, x_{10}, x_{18}\} &\subset \mathcal{P}(E_7); \\ \psi(\varrho_{11}) &= \delta_2(\zeta_5 \otimes \zeta_5) + x_8 \otimes \varrho_3; \\ \psi(\varrho_{15}) &= \delta_2(\zeta_9 \otimes \zeta_5) + \delta_3(\zeta_7 \otimes \zeta_7); \\ \psi(\varrho_{19}) &= \delta_2(\zeta_9 \otimes \zeta_9); \\ \psi(\varrho_{23}) &= \delta_2(\zeta_{17} \otimes \zeta_5) + \delta_3(\zeta_7 x_8 \otimes \zeta_7 - \zeta_7 \otimes \zeta_7 x_8); \\ \psi(\varrho_{27}) &= \delta_2(\zeta_{17} \otimes \zeta_9) - \delta_3(\zeta_7 \otimes \zeta_{19}); \\ \psi(\varrho_{35}) &= \delta_2(\zeta_{17} \otimes \zeta_{17}) + x_8 \otimes \varrho_{27} - \varrho_{27} \otimes x_8 + x_8 \otimes x_8 \varrho_{19}; \\ \psi_2(\zeta_{2i-1}) &= 0, i \in e(E_7, 2).\end{aligned}$$

Theorem 5. *With respect to the ring presentation*

$$H^*(E_8) = \Delta_{\mathbb{Z}}(\varrho_3, \varrho_{15}, \varrho_{23}) \otimes \Lambda_{\mathbb{Z}}(\varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47}, \varrho_{59}) \bigoplus_{p=2,3,5} \tau_p(E_8)$$

where

$$\tau_2(E_8) = \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}, x_{30}, \mathcal{C}_I]^+}{\langle x_6^8, x_{10}^4, x_{18}^2, x_{30}^2, \mathcal{D}_J, \mathcal{R}_K, \mathcal{S}_{I,J}, \mathcal{H}_{t,L} \rangle} \otimes \Delta_{\mathbb{F}_2}(\varrho_3, \varrho_{15}, \varrho_{23}) \otimes \Lambda_{\mathbb{F}_2}(\varrho_{27})$$

with $t \in e(E_8, 2) = \{3, 5, 9, 15\}$, $K, I, J, L \subseteq e(E_8, 2)$, $|I|, |J| \geq 2$, $|K| \geq 3$;

$$\begin{aligned}\tau_3(E_8) &= \frac{\mathbb{F}_3[x_8, x_{20}, \mathcal{C}_{\{4,10\}}]^+}{\langle x_8^3, x_{20}^3, x_8^2 x_{20}^2 \mathcal{C}_{\{4,10\}}, \mathcal{C}_{\{4,10\}}^2 \rangle} \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_{15}, \varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47}); \\ \tau_5(E_8) &= \frac{\mathbb{F}_5[x_{12}]^+}{\langle x_{12}^5 \rangle} \otimes \Lambda_{\mathbb{F}_5}(\varrho_3, \varrho_{15}, \varrho_{23}, \varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47}),\end{aligned}$$

and where

$$\begin{aligned}\varrho_3^2 &= x_6, \varrho_{15}^2 = x_{30}, \varrho_{23}^2 = x_6^6 x_{10}, \\ x_{2s} \varrho_{3s-1} &= 0 \text{ for } s = 4, 5; \\ x_8 \varrho_{59} &= x_{20}^2 \mathcal{C}_{\{4,10\}}; x_{20} \varrho_{23} = x_8^2 \mathcal{C}_{\{4,10\}}; \\ x_{12} \varrho_{59} &= 0,\end{aligned}$$

the reduced co-product ψ is given by

$$\begin{aligned}\{\varrho_3, x_6, x_8, x_{10}, x_{12}, x_{18}, x_{20}\} &\subset \mathcal{P}(E_8); \\ \psi(\varrho_{15}) &= \delta_2(\zeta_9 \otimes \zeta_5) + x_6^2 \otimes \varrho_3 - \delta_3(\zeta_7 \otimes \zeta_7) + x_{12} \otimes \varrho_3; \\ \psi(\varrho_{23}) &= \delta_2(\zeta_{17} \otimes \zeta_5) + \sum_{s+t=2} x_6^s \zeta_5 \otimes x_6^t \zeta_5 + x_{10}^2 \otimes \varrho_3 \\ &\quad + \delta_3(x_8 \zeta_7 \otimes \zeta_7 - \zeta_7 \otimes \zeta_7 x_8) - \delta_5(\zeta_{11} \otimes \zeta_{11}); \\ \psi(\varrho_{27}) &= \delta_2(\zeta_{17} \otimes \zeta_9) + \delta_3(\zeta_{19} \otimes \zeta_7) - x_{12} \otimes \varrho_{15} + (x_6^4 + 2x_{12}^2) \otimes \varrho_3; \\ \psi(\varrho_{35}) &= \delta_2(\zeta_{17} \otimes \zeta_{17}) - \varrho_{27} \otimes x_8 + x_8 \otimes \varrho_{27} + x_{20} \otimes \varrho_{15} + \delta_3(x_8 \zeta_{19} \otimes \zeta_7)\end{aligned}$$

$$\begin{aligned}
& +2x_{12} \otimes \rho_{23} + \delta_5(x_{12}\zeta_{11} \otimes \zeta_{11} + 3\zeta_{11} \otimes \zeta_{11}x_{12}); \\
\psi(\varrho_{39}) &= \delta_2\left(\sum_{s+t=2} x_{10}^s \zeta_9 \otimes x_{10}^r \zeta_9\right) - \delta_3(\zeta_{19} \otimes \zeta_{19}) + x_{12} \otimes \varrho_{27} \\
& +2x_{12}^2 \otimes \varrho_{15} - x_{12}^3 \otimes \varrho_3; \\
\psi(\varrho_{47}) &= \delta_2\left(\sum_{s+t=6} x_6^s \zeta_5 \otimes x_6^r \zeta_5\right) - x_{20} \otimes \varrho_{27} + \varrho_{39} \otimes x_8 + \delta_3(x_{20}\zeta_{19} \otimes \zeta_7) \\
& +2x_{12} \otimes \varrho_{35} + x_{12}^2 \otimes \varrho_{23} + \delta_5(\zeta_{11} \otimes x_{12}^2 \zeta_{11} + \sum_{s+t=2} x_{12}^s \zeta_{11} \otimes x_{12}^r \zeta_{11}); \\
\psi(\varrho_{59}) &= \delta_2(x_{10}^2 \zeta_{29} \otimes \zeta_9 + x_{30} \zeta_{17} \otimes \zeta_5 x_6 + x_{18} \zeta_{29} \otimes \zeta_5 x_6 + x_6^4 \zeta_{29} \otimes \zeta_5 \\
& + \zeta_{29} \otimes \zeta_{29} + x_{10}^2 \zeta_{17} \otimes \zeta_9 x_6^2 + \zeta_{17} \otimes x_6^2 \zeta_{29} + x_6^4 \zeta_{17} \otimes \zeta_5 x_6^2 \\
& + x_{18} \zeta_{17} \otimes \zeta_5 x_6^4 + x_6^4 x_{10}^2 \otimes \zeta_5 \zeta_9 + x_{10}^2 \otimes \zeta_9 \zeta_{29} + x_6^4 \otimes \zeta_5 \zeta_{29}) \\
& + \delta_3\left(\sum_{s+t=1} (-x_{20})^s \zeta_{19} \otimes x_{20}^r \zeta_{19}\right) + 2\delta_5\left(\sum_{s+t=4} (-x_{12})^s \zeta_{11} \otimes x_{12}^r \zeta_{11}\right); \\
\psi_p(\zeta_{2i-1}) &= 0 \text{ for } (p, i) = (2, 3), (2, 5), (2, 9), (3, 4), (3, 10), (5, 6); \\
\psi_2(\zeta_{29}) &= x_{10}^2 \otimes \zeta_9 + \zeta_{17} \otimes x_6^2 + x_6^4 \otimes \zeta_5.
\end{aligned}$$

Explanations on the notation used in Theorems 1–5 are in order.

i) Given a ring \mathcal{A} and a finite set $\{t_i\}_{1 \leq i \leq n}$ the symbols

$$\Delta_{\mathbb{Z}}(t_i)_{1 \leq i \leq n}, \Lambda_{\mathbb{Z}}(t_i)_{1 \leq i \leq n} \text{ and } \mathcal{A}[t_i]_{1 \leq i \leq n}^+$$

stand, respectively, for the \mathbb{Z} -module with basis the square free monomials in t_1, \dots, t_n , the exterior algebra generated by t_1, \dots, t_n over \mathbb{Z} and the polynomials ring in t_1, \dots, t_n over \mathcal{A} of positive degrees in t_1, \dots, t_n .

ii) The relations of the types $\mathcal{D}_J, \mathcal{R}_K, \mathcal{S}_{I,J}, \mathcal{H}_{t,I}$ (that occur only in the presentations of $\tau_2(E_7)$ and $\tau_2(E_8)$) are many. Their full expressions can be found in [4, Theorem 6].

Finally, a few words concerning the proofs of Theorems 1–5. Firstly, the presentation of $H^*(G)$ as a ring comes from [4, Theorem 6]. Next, granted with Lemmas 3.2 and 3.3 and taking into account for the obvious relation

$$\delta_p = r_p^{-1} \circ \beta_p : H^*(G \times G; \mathbb{F}_p) \rightarrow \tau_p(G \times G),$$

it is straightforward to apply (3.1) to deduce the expressions for $\psi(\varrho_i)$.

References

- [1] A. Borel, Homology and cohomology of compact connected Lie groups. Proc. Nat. Acad. Sci. U. S. A. 39(1953). 1142–1146.
- [2] A. Borel, Sur l’homologie et la cohomologie des groupes de Lie compacts connexes, Amer. J. Math. 76(1954), 273–342.
- [3] H. Duan, Xuezhong Zhao, The integral cohomology of complete flag manifolds, arXiv: math.AT/0801.2444.
- [4] H. Duan and Z. Zhao, Schubert calculus and the integral cohomology of simple Lie groups, arXiv: math.AT/ 0711.2541.

- [5] H. Duan, Xuezhi Zhao, Schubert calculus and the Hopf algebra structures of exceptional Lie groups, arXiv: math.AT/ 0903.4501.
- [6] V.G. Kač, Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups, Invent. Math. 80(1985), no. 1, 69–79.
- [7] J. Leray, Sur l’homologie des groupes de Lie, des espaces homogènes et des espaces fibré principaux, Colloque de topologie Algébrique, Bruxelles 1950, 101–115.
- [8] R. Marlin, Une conjecture sur les anneaux de Chow $A(G, \mathbb{Z})$ renforcée par un calcul formel, Effective methods in algebraic geometry (Castiglione, 1990)
- [9] H. Pittie, The integral homology and cohomology rings of $SO(n)$ and $Spin(n)$, J. Pure Appl. Algebra 73(1991), 105-153.
- [10] J. P. Serre, *Algèbre locale. Multiplicités*, Lecture Notes in Mathematics, vol.11, Springer-Verlag, Berlin-New York 1965.